Linear Algebra

Eigenvalue – Eigenvector

Department of Computer Engineering

Sharif University of Technology

Hamid R. Rabiee rabiee@sharif.edu
Maryam Ramezani maryam.ramezani@sharif.edu

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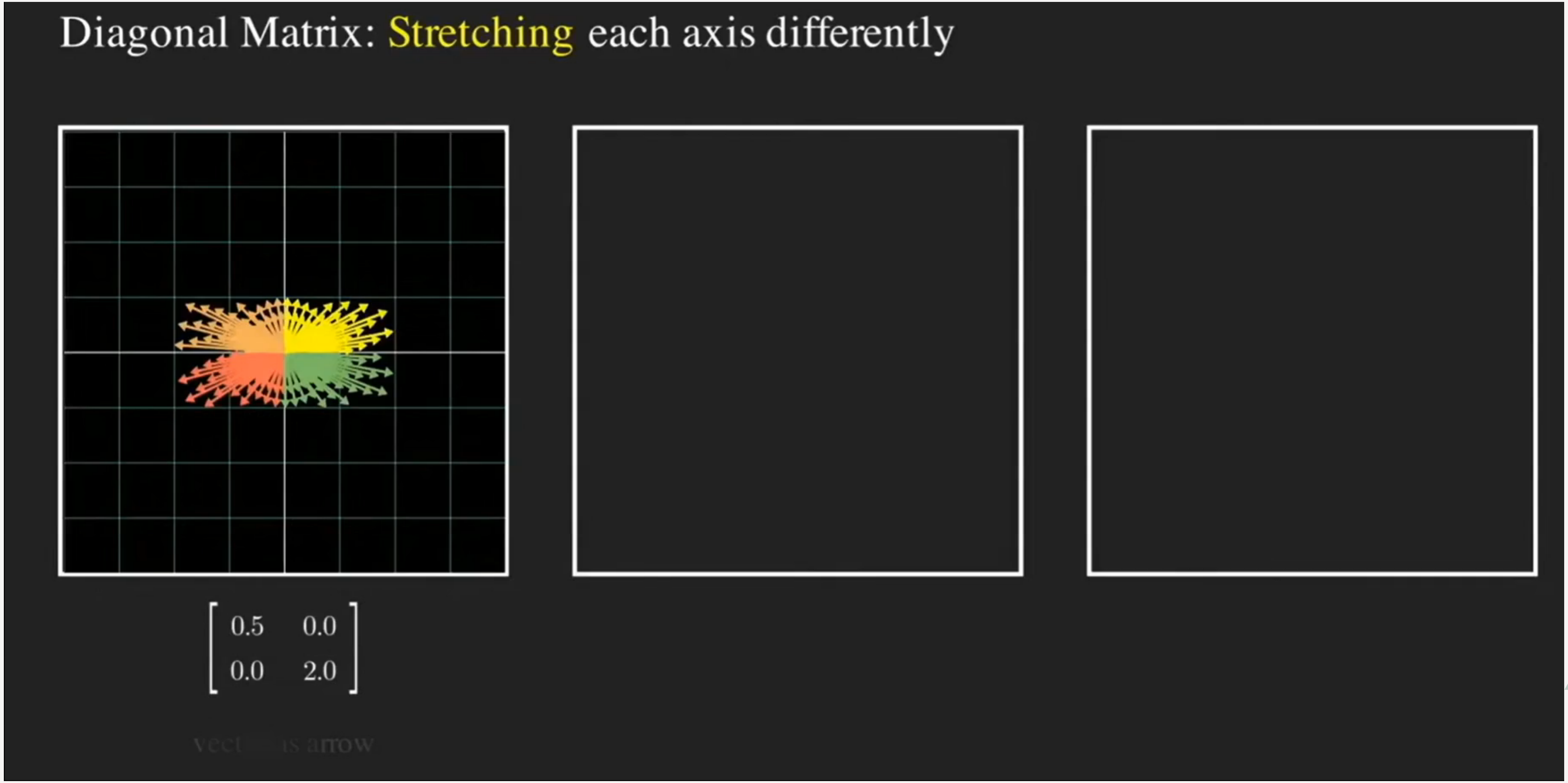
Introduction

Note

- In this slide, matrix is a square matrix!

Review


Diagonal Matrix: Stretching each axis differently
0.5 0.0
0.0 2.0

Orthogonal Matrix

Definition

- A matrix is orthogonal if its columns are:
  - orthogonal
  - has norm 1

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Eigenvector & Eigenvalue

Motivation

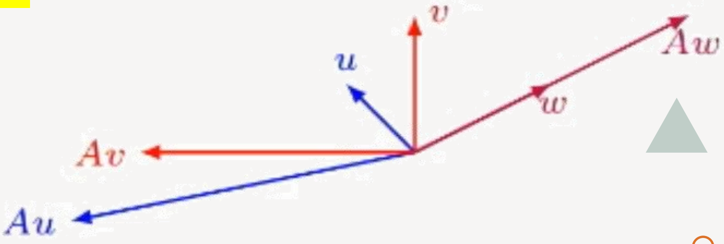
- $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$

  $u = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow Au = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$

  $v = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \Rightarrow Av = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$

  $w = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow Aw = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

- Vector “w” keeps the straight, but changes the scale.



Eigenvector & Eigenvalue

Definition

An **eigenvector** of a square $n \times n$ matrix A is nonzero vector v such that $Av = \lambda v$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution v of $Av = \lambda v$; such an v is called an *eigenvector corresponding to* λ.

- **An eigenvector must be nonzero, by definition, but an eigenvalue may be zero.**

Example

- $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\lambda = 2$
- Show that 7 is an eigenvalue of matrix B, and find the corresponding eigenvectors.

$$B = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

Eigenspace

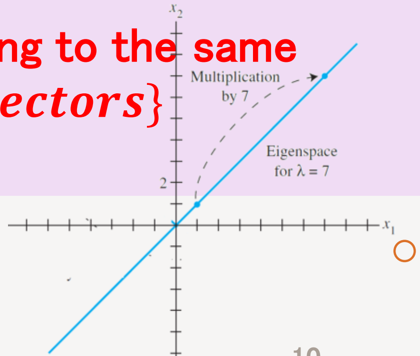
Note

λ is an eigenvalue of an $n \times n$ matrix:

$$Av = \lambda v \Rightarrow (A - \lambda I)v = 0$$

The set of all solutions of above is just the null space of the matrix $A - \lambda I$. So this set is the *subspace* of $\mathbb{R}^n$ and is called the **eigenspace** of A corresponding to λ . The eigenspace consists of the zero vector and all the eigenvectors corresponding to λ.

Eigenspace: A vector space formed by eigenvectors corresponding to the same eigenvalue and the origin point. $span\{corresponding\ eigenvectors\}$



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Characteristic Polynomial

Characteristic Polynomial

Note

- $Av = \lambda v \Rightarrow Av - \lambda vI = 0 \Rightarrow (A - \lambda I)v = 0 \;\; v \neq 0$
  - $v \in N(A - \lambda I)$
  - $A - \lambda I$ **must be singular.**
  - Proof that for finding the eigenvalue we should solve the determinate zero equation. Look at nullspace, rank and nullity theorem, singular matrix, and det zero!
- **Characteristic polynomial** $\det(A - \lambda \mathrm{I})$
- **Characteristic equation** $\det(A - \lambda \mathrm{I}) = 0$
- If λ is an eigenvalue of A, then the subspace E_λ = {span{v} | Av = λ v} is called the eigenspace of A associated with λ. (This subspace contains all the span of eigenvectors with eigenvalue λ, and also the zero vector.)
- **Eigenvector is basis for eigenspace.**
- Set of all eigenvalues of matrix is $\sigma(A)$ *named spectrum of a matrix*

Characteristic Polynomial

Note

- Instead of $\det(A-\lambda I)$, we will compute $\mathbf{det}(\boldsymbol{\lambda}\mathbf{I}-\boldsymbol{A})$. Why?
  - $\det(A-\lambda I)=(-1)^{\mathrm{n}}\det(\lambda I-\mathrm{A})$
  - Matrix $n\times n$ with real values has ⋯⋯ eigenvalues.

Finding Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix.

1. First, find the eigenvalues λ of A by solving the equation $\det(\lambda I - A) = 0$.
2. For each λ, find the basic eigenvectors $X \neq 0$ by finding the basic solutions to $(\lambda I - A) X = 0$.

To verify your work, make sure that $AX = \lambda X$ for each λ and associated eigenvector X.

Example

Find eigenvalues and eigenvectors, eigenspace (E), and *spectrum* of matrix $A = \begin{bmatrix} \mathbf{3} & \mathbf{-2} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}$:

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 = 0 \Rightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 2 \end{cases}$$

$$\left.\begin{matrix} \lambda_1 = 1 \\ (A - \lambda_1 I)q_1 = 0 \end{matrix}\right\} \Rightarrow q_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\left.\begin{matrix} \lambda_2 = 2 \\ (A - \lambda_2 I)q_2 = 0 \end{matrix}\right\} \Rightarrow q_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Eigenvalues={1,2}

Eigenvectors=$\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\}$

$E_1(A) = span\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$ $E_2(A) = span\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\}$

$\sigma(A)$={1,2}

$$\boldsymbol{AQ = Q\Lambda} \Rightarrow \begin{bmatrix} \mathbf{3} & \mathbf{-2} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{1} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{1} & \mathbf{1} \end{bmatrix}\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{2} \end{bmatrix}$$

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Properties

Expanding the Characteristic equation of A to polynomial form

Theorem (1)

To have (1) scalar for largest degree instead of $|\boldsymbol{A} - \boldsymbol{\lambda}\mathbf{I}|$, consider $|\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{A}|$

$$f(\lambda) = |\lambda I - A| = \lambda^n + c_{n-1}\lambda^{n-1} + \ldots + c_1\lambda + c_0$$

Proof?

- The n roots of this polynomial are eigenvalues!
  - $f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_n)$
- What is c_{n-1}?
  - $c_{n-1} = -trace(A)$
- What is c_0?
  - $c_0 = \det(-A) = (-1)^n \det(A)$

Sum and Product of eigenvalues

Theorem (2)

If A is an n × n matrix, then the sum of the n eigenvalues of A is the trace of A.

(coefficient c_{n-1} in expanded characteristic equation)

Other view: $f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n)$

$$|\lambda I - A| = \lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_1\lambda + c_0$$

Proof?

Theorem (3)

If A is an n × n matrix, then the product of the n eigenvalues is the determinant of A.

(coefficient c_0 in expanded characteristic equation)

Proof?

Determinant and Eigenvalue

Theorem (4)

$0 \in \sigma(A) \iff |A|=0$

Proof?

Conclusion: The Invertible Matrix Theorem

Let A be an $n \times n$ matrix. Then A is invertible if and only if:

- The number 0 is not an eigenvalue of A .
- The determinant of A is not zero.

An Important Theorem!

Theorem (5)

The eigenvalues of a triangular (upper/lower/diagonal) matrix are the entries on its main diagonal. For the diagonal matrix the eigenvectors are e_is. For upper /lower matrices, Q matrix of $\boldsymbol{AQ = Q\Lambda}$ will be upper/lower triangular matrix.

Proof?

Real Eigenvalues of different matrices

- Projection matrix
  - 0 , 1
  - If rank(P)=r with n columns, what are the repetition of the eigenvalues?
    - 0: n-r   1:r
- Reflection matrix
  - 1 , -1
- Permutation matrix
  - 1 , -1

Characteristic Equation

Example

Find the eigenvalues with their repetition and eigenvectors:

- ❑ $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- ❑ The characteristic polynomial of a 6×6 matrix is $\lambda^6 - 4\lambda^5 - 12\lambda^4$.
- ❑ $B = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$
- ❑ $C = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$
- ❑ $D = \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix}$

Eigenvalues of matrix products

Theorem (6)

The nonzero Eigenvalues of AB equal to the nonzero eigenvalues of BA.

Proof?